CNCM Online Round 2 Solutions

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✾ 1 Round 2 Solutions

Problem 1. We know that the chance that Adi makes exactly e buckets is $p^e(1-p)$, because he must make his first e shots then miss the next one. Expressing this for each $35 \leq e \leq 69$, we get the total probability as follows.

$$p^{35}(1-p) + p^{36}(1-p) + \cdots + p^{69}(1-p)$$

$$(1-p)(p^{35} + p^{36} + -\cdots + p^{69})$$

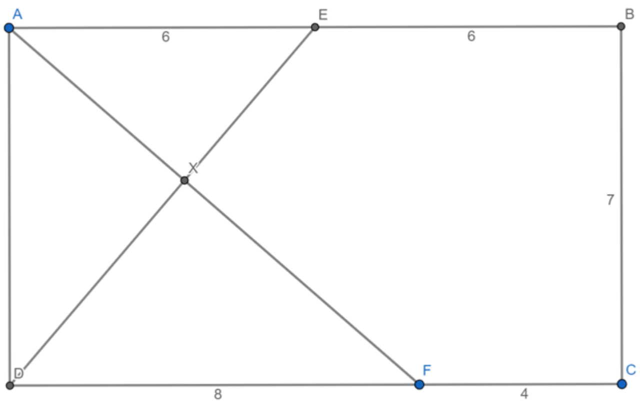
Using finite geometric formula (and noting $p \leq 1$), we reduce to as follows.

$$(1-p)(\frac{p^{35}}{1-p} - \frac{p^{70}}{1-p})$$

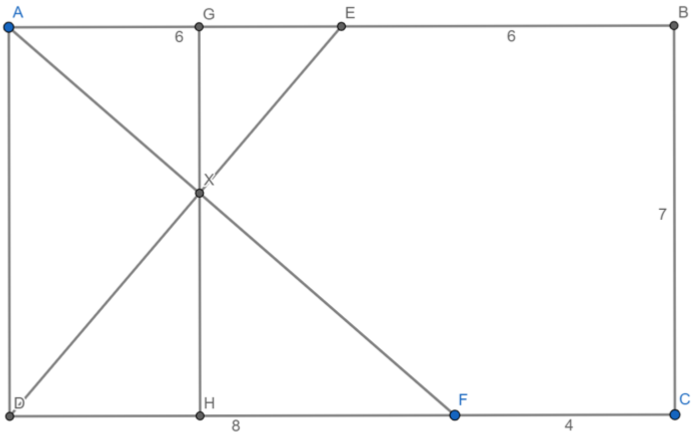
$$p^{35} - p^{70}$$

Substituting $x = p^{35}$, we want to maximize $x - x^2$. We see this is when $x = \frac{1}{2}$, or $p = \frac{1}{\sqrt[35]{2}}$. Our answer is thus $2 + 35 = \boxed{37}$

Problem 2. We want to complimentary count, since finding the area of the pentagon directly is challenging.



The areas of ΔADE and ΔADF are easy to evaluate, which are 21 and 28 respectively. We just want to find the area of ΔADX. Since we already have the base length, $AD = 7$, we want to find the height. We do this by dropping the altitudes from X to AB and CD.



Let G and H be the feet of the altitudes as labeled. Clearly, $\Delta AEX \sim \Delta FDX$, with a length ratio of $3:4$. Let $AG = 3x$. Then by our similarity, we have $GF = 4x$. We also have that $DG = 3x$. So, we see $3x + 4x = 8$, or $x = \frac{8}{7}$. We know the area of ΔADX is $\frac{1}{2} \cdot 7 \cdot 3x = 12$. We put this back into what we had earlier. The area of $AEXFD = 28 + 21 - 12 = 37$. So, the area of the rest of the rectangle, which is the area of $BCFXE$, is $84 - 37 = \boxed{47}$

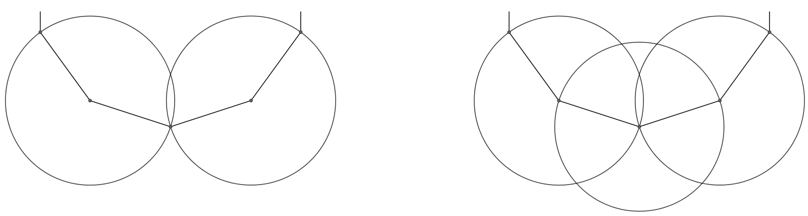
Problem 3. By the second condition, we know that $n = kp - 1$ for some integer k. If we substitute this into the first condition, we see that $(kp-1)^2 \equiv 1 \mod (p^2)$. We can expand this to find that $k^2p^2 - 2kp + 1 \equiv 1 \mod (p^2) \implies -2kp \equiv 0 \mod (p^2)$. Since p is odd, this means that p is a factor of k, so p^2 divides $n+1$. Now, we just test cases. It's not hard to see that $p = 3$ has 22 cases, $p = 5$ has 8 cases, $p = 7$ has 4, $p = 11$ has 1 case, and $p = 13$ has 1 case. Since $17^2 > 200$, the answer is $22 + 8 + 4 + 1 + 1 = \boxed{36}$.

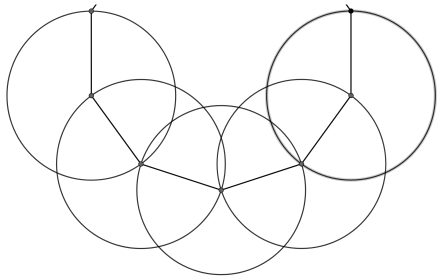
Problem 4. It can be fairly easily shown that the Slow Rook will never move down. This implies that the Slow Rook will move up 5 times, left 5 times, and right 5 times, which further implies that only the first 6 Sleeping Pawns matter, and we need to multiply our answer by 4^2 at the end to account for the possible positions of the last 2 Sleeping Pawns. Another important thing to note is that the Slow Rook can only pass between two adjacent Sleeping Pawns iff the Sleeping Pawns are at least 2 spaces apart vertically. Thus, we rephrase the problem as follows: How many 6-digit, base 4 numbers exist such that the first two digits have a difference of at least 2, and each pair of digits after have a difference of no more than 1? This is doable by recursion. In the following table, let each column be the number of digits in the number. Let each row represent first most digit. Let each table entry be the amount of numbers whose digits are no more than 1 apart with the conditions set by the rows and columns.

	1	2	3	4	5
0	1	2	5	13	34
1	1	3	8	21	55
2	1	3	8	21	55
3	1	2	5	13	34

To finish, the 6th we need to add can be a $1, 2, 3$, or 4. If it's 1, then the next digit must be 3 or 4. If it's 2, then the next digit can only be 4. Thus by symmetry, the answer is $4^2(2 \cdot ((55 + 34) + 34)) = \boxed{3936}$.

Problem 5.





Let (V_i) denote the circle centered at vertex V_i. Order V_i counterclockwise. For $n \geq 7$, the only (V_i) that intersect (V_k) are for $k \equiv i-2, i-1, i, i+1, i+2 \mod n$. This observation is motivated by considering the concurrency of all (V_i) for $n = 6$ and the tangency of (V_i), (V_{i+2}) at $n = \infty$ and can be shown rigorously via trigonometry. Oriented as shown in the first diagram, we know the other point of intersection of (V_1), (V_3) (which are the two circles shown) X lies within the polygon trivially (as long as interior angles exceed 90° as they must for $n \geq 7$ as (V_1) must stay on the same side of the tangent to (V_1) at V_2 at all times and the same must hold for V_3 and (V_3) which is only possible under V_2 when both tangents to either circle lies completely outside the polygon.) We can then draw in (V_2) practically freehand to get figure 2, and similarly draw in circles to get figure 3. The figure clearly shows that there are three regions that can be associated with each vertex and 2 regions that can be associated with each segment for a total of $5n+1$ regions, where the last region is the central region. For $n = 5$, the argument still holds (although the configuration is a bit different) as although the distance between V_1, V_4 is less than 2 this is accounted for exactly by V_4=V_{-1} (and it is notable that we can again group regions by associating 3 per vertex and 2 per line segment in a natural way.) For $n = 6, 4,$ and 3, the claim about which V_i intersect a given V_k is false (that is, some vertices are either omitted or double counted in the argument.) These cases must be checked manually for a grand total of $\sum_{n=7}^{27}(5n+1) + f(6) + [5(5)+1] + f(4) + f(3) = \boxed{1616}$.

Problem 6. Rewrite the equation as

$$y^2 - y(x^2 + 3x + 6) + (6x^2 + 6x) = 0.$$

Viewing the LHS as a function of y and using the quadratic formula gives

$$y = \frac{(x^2 + 3x + 6) \pm \sqrt{(x^2 + 3x + 6)^2 - 24x(x + 1)}}{2}.$$

Note $x^2 + 3x$ is always even, so we can simplify to

$$y = \frac{x^2 + 3x + 6}{2} \pm \sqrt{\left(\frac{x^2 + 3x}{2} + 3\right)^2 - 6x(x + 1)}.$$

Notice the expression in the square root is equal to

$$\left(\frac{x^2 + 3x}{2} - 3\right)^2 + 12x.$$

If it is less than $\left(\frac{x^2+3x}{2} - 2\right)^2$, then we know that the expression in the square root cannot be a perfect square as it will be in between two consecutive perfect squares. Thus, we rule out the all x that satisfy the following:

$$\left(\frac{x^2 + 3x}{2} - 2\right)^2 - \left(\frac{x^2 + 3x}{2} - 3\right)^2 > 12x.$$

Using the difference of squares and simplifying gives

$$x^2 - 9x - 5 > 0.$$

Solving the quadratic tells us we can rule out $x > 9$. Thus it remains to test $x = 1, 2, 3, 4, 5, 6, 7, 8, 9$ from which we can easily find that $x = 4$ gives us the desired solution of $(4, 30)$ corresponding to an answer of

$\boxed{34}$. **Problem 7.** (Solution by David Altizio, on his blog here, with his diagram omitted)

First observe that $XA \perp AD$, so $\triangle XAP$ is a 45-45-90 triangle. The fact that M is the midpoint of $\overline{CD}$ implies that $\angle MAD = \angle MPA = 45°$. Combined with $\angle MCA = \angle MDP$ and $\angle CMD = 90°$, we deduce $\triangle MCD$ is a 45-45-90 triangle as well. Now the $XY \parallel CD$ condition implies

$$\angle YXB = \angle(BX, CD) = \frac{\widehat{MC} + \widehat{BD}}{2} = \frac{\widehat{MD} + \widehat{BD}}{2} = \angle MAB. \qquad (*)$$

In turn, letting O be the center of the circle, $\triangle AOM \sim \triangle XBY$.

Now construct point Q so that $AXQP$ is a square.

Since M is the midpoint of $\overline{AQ}$ and O is the midpoint of $\overline{AB}$, $MO \parallel QB$. This means triangles ABQ and XBY are similar isosceles triangles. But, actually, they are spirally similar; the dual spiral similarity implies $\triangle BAX \sim \triangle BQY$. The condition $BX = BY$ yields that, in fact, these triangles are congruent, so $QY = AX = QX = QP$; in turn, Q is the circumcenter of $\triangle PXY$.

Finally, observe that $\angle XYP = \angle QYB = 45°$ – the former from $\angle XQP = 90°$, the latter from the spiral similarity – so

$$\angle QXY = \angle XYQ = \angle PYB = 10°.$$

But this when combined with $(*)$ yields $\angle DAB = 10°$ as well, so

$$\angle XCM = \angle MBA = 45° - \angle PAB = \boxed{35°}.$$